Visakha Vidyalaya - Colombo 05
Final Term Test -2024 October
Combined Mathematics I

Grade 13

Part B

(11) (a) Show that the equation $kx^2 + 2x + k = 0$ has real and distinct roots when $0 < k < 1$.

Let above roots be α and β ($<\alpha$) prove that both roots are negative.

Find the value of $(1+\alpha)(1+\beta)$ interms of k and hence deduce that $-1 < k < 0$ and $\beta < -1$.

Show that $|1 + \alpha| + |1 + \beta| = \frac{2}{k}\sqrt{1 + k^2}$.

Show also that the quadratic equation with roots $|1+\alpha|$ and $|1+\beta|$ is

$kx^2 - 2\sqrt{1 + k^2}\, x + 2(1 - k) = 0$.

(b) $f(x)$ is a polynomial of degree 2 or more . Show that $(x - r)^2$ is a factor of $f(x)$ if and only if $f(r) = f'(r) = 0 (r \in \mathbb{R})$.

(i) let $g(x) = x^3 + ax^2 + bx + c$. If a, b, c are real constants and $a^2 < 3b$, show that $(x - r)^2$ is not a factor of $g(x)$.

(ii) Let $h(x) = x^3 - 3x + k$ where k is a real constant. When $(x - r)^2$ is a factor of $h(x)$ show that $k = \pm 2$. Find all the factors of $h(x)$ for that values of k.

(12) (a) When the department of motor traffic issues number plates, it uses three letters from the English alphabet and then four digits.

If letter C is used as the first letter for cars,

(i) How many distinct number plates can be issued for cars?

(ii) Among the above number plates how many are odd numbers whose first and last digits are the same?

(b) Let $U_r = \frac{r^2+5r+2}{r^2(r+1)^2}$ and $f(r) = \frac{1}{r}$ for $r \in \mathbb{Z}^+$

Determine the values of the constants A and B such that

$U_r = A[f(r) - f(r+1)] + B[f(r^2) - f(r+1)^2]$ for $\in \mathbb{Z}^+$.

Hence show that $\sum_{r=1}^{n} U_r = 3 - \frac{n+3}{(n+1)^2}$.

Is the series $\sum_{r=1}^{\infty} U_r$ convergent?

Find the minimum integral value of n such that $S_\infty - S_n < \frac{1}{2}$

where $S_n = \sum_{r=1}^{n} U_r$ and $S_\infty = \sum_{r=1}^{\infty} U_r$.

(13) (a) Let $X = \begin{pmatrix} a & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$ $Y = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -2 & b \end{pmatrix}$ and $Z = \begin{pmatrix} 6 & -5 \\ c & -3 \end{pmatrix}$.

(i) If $c \neq \frac{18}{5}$ Show that Z^{-1} exist.

(ii) Find the values of the constants a, b and c such that $YX^T = Z + 2I$ where I is the unit matrix of order 2.

(iii) For the above value of c write down Z^{-1}

(iv) Find the matrix P such that $Z^{-1}PZ = Z^2 + 7Z$.

(b) Define the modulus of Z, |Z| and the complex conjugate $\overline{Z}$ for $Z \in \mathbb{C}$.

Show that $|Z|^2 = Z\,\overline{Z}$ and $Z + \overline{Z} = 2\text{Re}Z$.

(i) For $Z_1, Z_2 \in \mathbb{C}$

Show that $|2Z_1 + 3Z_2|^2 = 4|Z_1|^2 + 12Re(Z_1\overline{Z_2}) + 9|Z_2|^2$.

Write down a similar expression for $|2Z_1 - 3Z_2|^2$

Hence show that $|2Z_1 + 3Z_2|^2 + |2Z_1 - 3Z_2|^2 = 8|Z_1|^2 + 18|Z_2|^2$

(ii) Describe the locus of the point represented by the complex number

$\text{Arg}(Z - 3) - Arg\,(Z + 3) = \frac{\pi}{4}$, and draw rough sketch of the locus of Z.

(iii) State De Moivre's theorem for a positive integral index.

Prove that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \sin\theta + i\cos\theta$

Hence show that $\left(1 + \sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right)^{12} + \left(1 + \sin\frac{\pi}{12} - i\cos\frac{\pi}{12}\right)^{12} = 0$

(14) (a) Let $f(x) = \dfrac{2x^2+x-1}{(x-2)^2}$ for $x \neq 2$.

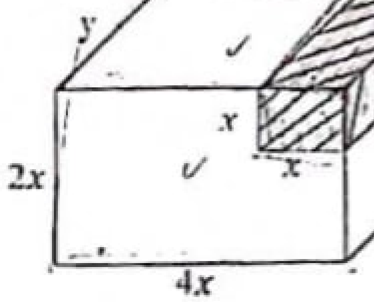
Show that $f'(x)$ the derivative of $f(x)$ is given by $f'(x) = \dfrac{-9x}{(x-2)^3}$ for $x \neq 2$.

Hence find the interval on which $f(x)$ is increasing and the interval on which $f(x)$ is decreasing. Also find the co-ordinates of the turning point of $f(x)$.

It is given that $f''(x) = \dfrac{18(x+1)}{(x-2)^4}$ for $x \neq 2$. Find the co-ordinates of the point of inflexion the graph $y = f(x)$.

Sketch the graph of $y = f(x)$ indicating the asymptotes, the turning point and the point of inflexion.

(b) The length width and the height of a cuboid log are $4x, 2x$ and y respectively. When a small cuboid of length breadth and height x, x and y respectively is removed as shown in the figure, the volume of the remaining part is ~~1764 cm^3~~. 3528



Show that the area of the remaining part of the log is given by $A=(14x^2+12xy)\ cm^2$

Find the value of x when A is minimum.

(15) (a) Using the substitution $y = (x + 2)$ express $\frac{3x^3+2}{(x+2)^3}$

In the form $\frac{3x^3+2}{(x+2)^3} = A + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$.

Where A, B, C and D are constants to be determined.

Hence find $\int \frac{3x^3+2}{(x+2)^3} dx$.

(b) Using integrating by parts, find $\int \cos 2x \ln \left(\frac{\cos x+\sin x}{\cos x-\sin x}\right) dx$

(c) Using a suitable substitution or otherwise, $\int_0^{2a} \frac{x^2}{(4a^2+x^2)^2} dx$.

(d) Using the relation

$\int_0^a f(x)dx = \int_0^a f(a-x)\, dx$ for a constant a show that

$I = \int_0^{\pi} \frac{x^2 \sin x}{(2x-\pi)(1+\cos^2 x)} dx = \frac{\pi}{2}\int_0^{2a} \frac{\sin x}{1+\cos^2 x} dx$

Hence show that $I = \frac{\pi^2}{4}$.

(16) Show that the perpendicular distance from the point (x_0, y_0) to the line $lx + my + n = 0$ is $\frac{|lx_0+my_0+n|}{\sqrt{l^2+m^2}}$.

Show that the straight line $(1-t^2)(x-a) + 2t(y-b) = r(1+t^2)$ touches the circle $(x-a)^2 + (y-b)^2 = r^2$ for all $t \in \mathbb{R}$

Let $S_1 \equiv 5x^2 + 5y^2 - 6x + 8y - 35 = 0$ and $S_2 \equiv x^2 + y^2 - 2x - 4y - 11 = 0$ show that S_1 and S_2 intersect each other and find the location of the centre of $S_2=0$ with respect to the circle $S_1=0$. Show that two chords of $S_1=0$ of length 4 units can be drawn touching the circle $S_2=0$ and find the equations of the two chords.

(17) (a) Write down tan (A+B) interms of tan A and tan B.

Show that $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)= tan\theta + sec\theta$.

Hence find the values of $\tan\frac{\pi}{12}$ and $\tan\frac{7\pi}{12}$.

(b) State the sine rule for any triangle ABC in the usual notation.

In the triangle ABC, $B\hat{A}C = \frac{\pi}{2}$. D and E are two points on BC such that BD =DE=EC

and angle $C\hat{A}E = \alpha$. Show that $tan\alpha = \sqrt{\frac{a^2-b^2}{4b^2}}$.

(c) (i) Solve the equation $2cos\theta(cos\theta - \sqrt{3}sin\theta) = 1$.

(ii) Solve the equation $tan(cos^{-1}x) = sin\left(cot^{-1}\left(\frac{1}{2}\right)\right)$.

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